LEBANESE AMERICAN UNIVERSITY
Department of Computer Science and Mathematics

Calculus III

Exam I

Spring 2009 (March 25, 2009)

Name: Solutions **ID:**

Circle the name of your instructor:

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Question Number	Grade
1. 28%	
2. 48%	
3. 24%	
Total	

1. (28%) Evaluate the following integrals:

(a) $\int \frac{\ln x}{x((\ln x)^2+1)}dx$ Let $u = \ln x$ $du = \frac{dx}{x}$

$$= \int \frac{u \, du}{u^2+1} = \frac{1}{2} \ln(u^2+1) + C$$

$$= \frac{1}{2} \ln(\ln^2 x + 1) + C$$

(b) $\int \frac{e^t}{e^{2t}-3e^t-10}dt$ Let $u = e^t \Rightarrow du = e^t dt$

$$\int \frac{du}{u^2-3u-10} = \int \frac{du}{(u-5)(u+2)} = \int \frac{A}{u-5} + \frac{B}{u+2} du$$

$A = 1/7$

$B = -1/7$

$$= A \ln|u-5| + B \ln|u+2| + C$$

$$= \frac{1}{7} \ln\left|\frac{u-5}{u+2}\right| + C = \frac{1}{7} \ln\left|\frac{e^t-5}{e^t+2}\right| + C$$

(c) $\int \frac{x^4}{x^2-4}dx$ Divide:

$$\frac{x^4}{x^2-4} = x^2\left(\frac{x^2-4+4}{x^2-4}\right)$$

$$= x^2\left(1 + \frac{4}{x^2-4}\right)$$

$$= x^2 + \frac{x^2}{x^2-4} = x^2 + \frac{x^2-4+4}{x^2-4}$$

$$= x^2 + 1 + \frac{4}{x^2-4}$$

$$= \int x^2 + 1 + \frac{4}{(x-2)(x+2)} dx$$

$$= \frac{x^3}{3} + x + 4\int \frac{A}{x-2} + \frac{B}{x+2} dx$$

$$= \frac{x^3}{3} + x + 4A \ln|x-2| + 4B \ln|x+2| + C$$

$$= \boxed{\frac{x^3}{3} + x + \ln\left|\frac{x-2}{x+2}\right| + C}$$

$$\frac{}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{}$$ $A = 1/4$ $B = -1/4$

(d) $\int \underbrace{x^2}_{u} \underbrace{\sinh x dx}_{dv}$ By parts (twice) $\boxed{\begin{array}{ll} u = x^2 & du = 2x\, dx \\ dv = \sinh x & v = \cosh x \end{array}}$

$= x^2 \cosh x - 2\int x \cosh x \, dx$ $\boxed{\begin{array}{ll} u = x & du = dx \\ dv = \cosh x & v = \sinh x \end{array}}$

$= x^2 \cosh x - 2\left[x \sinh x - \int \sinh x \, dx\right] + c$

$= x \cosh x - 2x \sinh x + 2 \cosh x + c$

2. (48%)Check if the following improper integrals converge or diverge.

(a) $\int_1^\infty \frac{x}{(x^3+1)^2} dx \approx \int_1^\infty \frac{x}{x^6} dx = \int_1^\infty \frac{1}{x^5} dx$

$\Rightarrow$ converges (p-int : $p > 1$)

(b) $\int_{-\infty}^\infty \frac{dx}{e^{(x^2)}} = 2\int_0^\infty \frac{dx}{e^{x^2}}$ (since it is an even function).

$= 2\left[\underbrace{\int_0^1 \frac{1}{e^{x^2}} dx}_{\text{proper}} + \int_1^\infty \frac{1}{e^{x^2}} dx\right]$.

Now $\int_1^\infty \frac{1}{e^{x^2}} dx < \underbrace{\int_1^\infty \frac{1}{x^2} dx}_{\text{conv. } p\text{-int } p>1}$ since $e^{(x^2)} > x^2 \Rightarrow \frac{1}{e^{x^2}} < \frac{1}{x^2}$.

$\Rightarrow \int^\infty \frac{1}{e^{x^2}} dx$ converges by DCT $\Rightarrow$ the whole int. also converges

(c) $\int_{\pi/2}^{\infty} \frac{3-\sin x}{x} dx$

$\sin x < 1 \Rightarrow -\sin x > 1$
$\Rightarrow 3 - \sin x > 2$

$\therefore \frac{3-\sin x}{x} > \frac{2}{x}$.

$$\therefore \int_{\pi/2}^{\infty} \frac{3-\sin x}{x} dx > \underbrace{\int_{\pi/2}^{\infty} \frac{2}{x} dx}_{\text{Diverges p-int. } p=1} \Rightarrow$$

$\Rightarrow$ The original <u>diverges</u> by DCT.

(d) $\int_{1}^{\infty} \frac{e^x}{\sqrt{1+x^2}} dx$

$e^x >> \sqrt{1+x^2} \Rightarrow$

$$\frac{e^x}{\sqrt{1+x^2}} \xrightarrow[x\to\infty.]{} \infty$$

$\therefore \int_1^{\infty} \frac{e^x}{\sqrt{1+x^2}} dx$ also diverges

since $\int_1^{\infty} f(x)\, dx$ diverges if $f(x). \longrightarrow \infty$

(e) $\int_{1}^{\infty} \frac{\ln x dx}{x}$

$\frac{\ln x}{x} > \frac{1}{x} \Rightarrow$

$> \int_1^{\infty} \frac{1}{x} dx$ (div. (p - int $p = 1$))

$\Rightarrow$ The original integral diverges also by DCT

(f) $\int_3^\infty \frac{dx}{x\sqrt{\ln x}}$ $= \lim_{t\to\infty} \int_3^t \frac{dx}{x\sqrt{\ln x}}$ $u = \ln x \Rightarrow du = \frac{dx}{x}$

$$\left[\int \frac{1}{x\sqrt{\ln x}}\,dx = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = 2\sqrt{u} = 2\sqrt{\ln x}.\right]$$

$$\therefore \lim_{t\to\infty}\left(2\sqrt{\ln x}\,\Big|_3^t\right) = 2\sqrt{\ln\infty} - 2\sqrt{\ln 3} = \infty$$

$\Rightarrow$ Integral Diverges

3. (24%) Evaluate the following improper integrals.

(a) $\int_{-\infty}^{\infty} \frac{x^3dx}{e^{(x^4)}}$ this function $f(x) = \frac{x^3}{e^{x^4}}$ is odd.

$\Rightarrow$ We will consider $\int_0^\infty \frac{x^3\,dx}{e^{x^4}}$: In case this converges $\Rightarrow$ to L $\Rightarrow$ the original will converge to $\underset{=0}{L-L}$

In case it diverge $\Rightarrow$ the original diverge too.

Consider $\int_0^\infty \frac{x^3}{e^{x^4}}\,dx$. Let $u = x^4 \Rightarrow du = 4x^3dx$.

$$= \lim_{t\to\infty}\left(\int_0^t \frac{x^3}{e^{x^4}}\,dx\right) = \lim_{t\to\infty} \frac{1}{4}\int e^{-u}\,du$$

$$= \lim_{t\to\infty} \frac{1}{4}\left(-e^{-x^4}\right)\Big|_0^t = \frac{1}{4}e^{-\infty} + \frac{1}{4}e^0 = \frac{1}{4}$$

$\Rightarrow L = 1/4 \Rightarrow$ limit $= L - L = 0$

(b) $\int_0^\infty xe^{-(x^2)}dx$ Let $u = x^2$ $du = 2x\,dx$.

$$= \lim_{t\to\infty} \int_0^t x e^{-x^2} dx = \lim_{t\to\infty} \int \frac{e^{-u}\,du}{2} = \lim -\frac{1}{2} e^{-u}$$

$$= \lim_{t\to\infty} \left(-\frac{1}{2} e^{-x^2} \Big|_0^t \right) = 1/2$$

$\Rightarrow$ Converges to $1/2$

(c) $\int_4^\infty \frac{2dx}{x\sqrt{x^2-4}}$ of the form $\int \frac{2\,dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)$

$= 2\cdot\frac{1}{2}\sec^{-1}(x/2)$

$= \sec^{-1}(x/2)$

$$= \lim_{\substack{t\to\infty \\ s\to 4}} \int_s^t \frac{2\,dx}{x\sqrt{x^2-4}} =$$

$$= \lim_{\substack{t\to\infty \\ s\to 4}} \cancel{2}\cdot\frac{1}{\cancel{2}}\sec^{-1}(x/2) = \left(\sec^{-1}\infty - \sec^{-1}2\right) = \boxed{\alpha - \beta}$$

Note: (1) If $\sec^{-1}\infty = \alpha \Rightarrow \sec\alpha = \infty \Rightarrow \frac{1}{\cos\alpha} = \infty \Rightarrow \cos\alpha = 0 \Rightarrow \boxed{\alpha = \pi/2}$

$\therefore \sec^{-1}\infty = \pi/2$

(2) If $\sec^{-1}2 = \beta \Rightarrow \sec\beta = 2 \Rightarrow \frac{1}{\cos\beta} = 2 \Rightarrow \cos\beta = 1/2 \Rightarrow \boxed{\beta = \pi/3}$

$\Rightarrow \sec^{-1}2 = \pi/3$

$\Rightarrow$ Integral converges to: $\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$